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Deterministic Computer Physics

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Completely arithmetic formulations, which possess exactly the same conservation laws and symmetry as their continuum counterparts, are given for both Newtonian and special relativistic mechanics. Applications are made to new models of fluid flow, vibration, diffusion, planetary evoultion, biological self-reorganization, and relativistic oscillation. Computer examples are described and discussed.

1. INTRODUCTION

It is somewhat startling that the foundations of both Newtonian and special relativistic physics can be reformulated using only *arithmetic*, with the very same conservation laws and symmetry following as in continuum physics. Moreover, not only does the arithmetic approach simplify the tools necessary for such theoretical considerations, but, from it, new types of models of natural phenomena follow readily and in a reasonable way. In this paper we will explore both the theory and the application of the arithmetic approach, which is motivated by, and implemented through, modern high-speed digital computer technology.

2. GRAVITY

It is always difficult to know how to begin correctly, so let us develop some intuition first by studying the following simple experiment with a force with which we are all familiar, namely, gravity.

If a particle of mass m, situated h feet above ground, is dropped from a position of rest, one can approximate its height x above ground every Δt seconds as it falls. For example, if $h=400$ and if one has a camera whose shutter time is Δt, then one can take pictures of the fall at the times $t_k = k\Delta t$, $k=0,1,2,\ldots$, and, from the photographs and the knowledge that

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$h=400$, approximate the heights $x(t_k)=x_k$ above ground by simple ratio and proportion. Suppose, then, that this has been done for $\Delta t=1.0$, that is, for a very slow camera, and that, to the nearest foot, one finds

$$x_0=400, \quad x_1=384, \quad x_2=336, \quad x_3=256, \quad x_4=144, \quad x_5=0$$

These data are recorded in column A of Table I.

Since it is always convenient mathematically to know how far a particle has traveled from its initial position, we first rewrite our data as

$$x_0=400-0, \quad x_1=400-16, \quad x_2=400-64$$

$$x_3=400-144, \quad x_4=400-256, \quad x_5=400-400$$

in which each term preceded by a negative sign is the distance traveled in time t_k. Each of these terms, however, is seen readily to have a factor of 16, so that we may rewrite our data next as

$$x_0=400-16(0), \quad x_1=400-16(1), \quad x_2=400-16(4)$$

$$x_3=400-16(9), \quad x_4=400-16(16), \quad x_5=400-16(25)$$

But, each term in parentheses is a perfect square, so that we now have

$$x_0=400-16(0)^2, \quad x_1=400-16(1)^2, \quad x_2=400-16(2)^2$$

$$x_3=400-16(3)^2, \quad x_4=400-16(4)^2, \quad x_5=400-16(5)^2$$

Finally, since $\Delta t=1$, we note that $t_0=0$, $t_1=1$, $t_2=2$, $t_3=3$, $t_4=4$, $t_5=5$, which implies

$$x_0=400-16(t_0)^2, \quad x_1=400-16(t_1)^2, \quad x_2=400-16(t_2)^2$$

$$x_3=400-16(t_3)^2, \quad x_4=400-16(t_4)^2, \quad x_5=400-16(t_5)^2$$

TABLE I. Velocity and Acceleration Calculations

Time	A Measured height	B Velocity by calculus	C Acceleration by calculus	D Velocity by arithmetic	E Acceleration by arithmetic
$t_0=0$	$x_0=400$	$v_0=0$	$a_0=-32$	$v_0=0$	$a_0=-32$
$t_1=1$	$x_1=384$	$v_1=-32$	$a_1=-32$	$v_1=-32$	$a_1=-32$
$t_2=2$	$x_2=336$	$v_2=-64$	$a_2=-32$	$v_2=-64$	$a_2=-32$
$t_3=3$	$x_3=256$	$v_3=-96$	$a_3=-32$	$v_3=-96$	$a_3=-32$
$t_4=4$	$x_4=144$	$v_4=-128$	$a_4=-32$	$v_4=-128$	$a_4=-32$
$t_5=5$	$x_5=0$	$v_5=-160$	$a_5=-32$	$v_5=-160$	

or, more succinctly,

$$x_k = 400 - 16(t_k)^2, \qquad k = 0,1,2,3,4,5 \tag{1}$$

Our problem next is how to proceed with (1). From the continuum point of view, one interpolates and extrapolates to yield the classical formula

$$x = 400 - 16t^2 \tag{2}$$

which is now amenable to the full power of the calculus. In this fashion one has, almost immediately,

$$v = -32t \tag{3}$$

$$a = -32 \tag{4}$$

Using (3) and (4), we have recorded in column B of Table I the particle's velocities, and in C the accelerations, at the times t_0, t_1, t_2, t_3, t_4, t_5 during its fall.

If our thinking, however, had been fashioned in a computer-dominated environment, then, because the real number system is not the number system of any computer, we might have chosen to store (1) in a memory bank and then tried to develop simple algebraic formulas for velocity and acceleration. We will show now how this can be done in such a fashion that the results are identical with those obtained from (2).

At the times that the pictures were taken, let P's velocity be denoted by $v_k = v(t_k)$, $k = 0,1,2,3,4,5$. Since P was dropped from a position of rest, let

$$v_0 = 0 \tag{5}$$

For $k > 0$, let v_k be defined as an *average* rate of change of position with respect to time by

$$\frac{v_{k+1} + v_k}{2} = \frac{x_{k+1} - x_k}{\Delta t}, \qquad k = 0,1,2,3,4 \tag{6}$$

The left-hand side of (6) is, of course, a smoothing operator, which is perfectly reasonable when dealing with experimental data. However, (6) is not as convenient from the computer viewpoint as is its equivalent form

$$v_{k+1} = -v_k + \frac{2}{\Delta t}(x_{k+1} - x_k), \qquad k = 0,1,2,3,4 \tag{6'}$$

which is a recursion formula. Substitution of $k = 0,1,2,3,4$ into (6′) yields, in order, $v_1=-32$, $v_2=-64$, $v_3=-96$, $v_4=-128$, $v_5=-160$, which are recorded in column D of Table I and are identical to the entries in column B.

Next, from a deterministic viewpoint, one would know a particle's initial position and velocity, but not its initial acceleration. The acceleration is intimately related to the force, which is at present under study. Thus, a_0 is not known and must be generated by some formula. If $a(t_k)=a_k$, we assume simply that

$$a_k=\frac{v_{k+1}-v_k}{\Delta t}, \qquad k=0,1,2,3,4 \tag{7}$$

From (7) and the values of v just found, we have $a_0=a_1=a_2=a_3=a_4=-32$, which are recorded in column E of Table I and are identical with the corresponding entries in column C. Formula (7) does not allow a determination of a_5, because this would require knowing v_6. Nevertheless, the entries indicate quite clearly that the acceleration due to gravity is constant with the value -32.

Now, just because the arithmetic formulas (5)–(7) have given the same results as the continuous formulas (3)–(4) does not mean that we have a formulation which has physical significance, since physics is characterized by conservation laws and symmetry. Surprisingly enough, our approach to gravity will also yield conservation and symmetry (Greenspan, 1973; 1980a). We will, however, for simplicity, confine attention here only to the conservation of energy.

We recall now the fundamental Newtonian dynamical equation

$$F=ma \tag{8}$$

the kinetic energy formula

$$K=\tfrac{1}{2}mv^2 \tag{9}$$

and, for a falling body with a $=-32$, the potential energy formula

$$V=32mx \tag{10}$$

The classical energy conservation law is simply

$$K_t+V_t\equiv K_0+V_0, \qquad t>0 \tag{11}$$

However, the data in column A of Table I were obtained from a sequential set of photographs, that is, at distinct times $t_k=k\Delta t$, so that in place of

(8)–(10) we can only assume

$$F_k = ma_k, \qquad k = 0, 1, 2, \ldots \tag{12}$$

$$K_k = \tfrac{1}{2} m v_k^2, \qquad k = 0, 1, 2, \ldots \tag{13}$$

$$V_k = 32 m x_k, \qquad k = 0, 1, 2, \ldots \tag{14}$$

Next, define work W_n, $n = 1, 2, 3, \ldots$, by

$$W_n = \sum_{i=0}^{n-1} (x_{i+1} - x_i) F_i \tag{15}$$

Then, by (5), (7) and (12)

$$\begin{aligned} W_n &= m \sum_{i=0}^{n-1} (x_{i+1} - x_i)\left(\frac{v_{i+1} - v_i}{\Delta t}\right) \\ &= \tfrac{1}{2} m \sum_{i=0}^{n-1} (v_{i+1} + v_i)(v_{i+1} - v_i) \\ &= \tfrac{1}{2} m v_n^2 - \tfrac{1}{2} m v_0^2 \end{aligned}$$

so that

$$W_n = K_n - K_0, \qquad n = 1, 2, 3, \ldots \tag{16}$$

which, incidentally, is independent of the structure of F and is a fundamental result in continuum mechanics. On the other hand, since $a_k = -32$, one has from (12) and (15) that

$$W_n = -32 m \sum_{i=0}^{n-1} (x_{i+1} - x_i) = -32 m x_n + 32 m x_0$$

so that, from (14),

$$W_n = -V_n + V_0, \qquad n = 1, 2, 3, \ldots \tag{17}$$

Finally, elimination of W_n between (16) and (17) yields

$$K_n + V_n \equiv K_0 + V_0, \qquad n = 1, 2, 3, \ldots \tag{18}$$

in complete analogy with (11). Moreover, since K_0 and V_0 are determined entirely from the initial data x_0 and v_0, it follows that K_0 and V_0 are the same in both (11) and (18), so that our arithmetic approach conserves *exactly* the same total energy, *independently* of Δt, as does classical Newtonian theory.

3. COMPUTER MODELING

Before extending the ideas of Section 2 to forces which are more complex than gravity, we must interpose some remarks about the way in which we will model.

Our primary aim at present is to simulate the actual forces, in the way they occur, in phenomena relating to solids, liquids, and gases. These forces are of two types, the *long-range* forces, like gravity and gravitation, which act on all atoms and molecules of a given substance, and the *local* forces, which are those which occur between each atom and its immediate neighbors only. Classically, the local forces are of the following general nature. If, for example, two molecules are pushed together then they repel, if pulled apart then they attract, and repulsion is of a greater order of magnitude than is attraction. A typical force formula for such an interaction might have a magnitude given by

$$F=-\frac{1}{r^8}+\frac{1}{r^{12}} \tag{19}$$

Our modeling procedure, then, will be as follows. A given solid, liquid, or gas will be represented by a finite set of particles. To each particle, two types of forces will be applied: (a) a long-range force, like gravity or gravitation, which will apply uniformly to all particles, and (b) a local force, similar to (19), which acts only between each particle and its immediate neighbors. Since the number of particles which can be handled effectively on a computer will be far less than the number of atoms and molecules in any interaction, we will simply decrease the exponents in (19) to compensate.

4. EXTENSIONS

For $\Delta t>0$, let $t_k=k\Delta t$, $k=0,1,2,\ldots$. Consider now a system of particles P_i, $i=1,2,3,\ldots,n$. Let P_i have mass m_i, and, at time t_k, be located at $\mathbf{r}_{i,k}$ with velocity $\mathbf{v}_{i,k}$ and acceleration $\mathbf{a}_{i,k}$. In analogy with (6) and (7), we assume that

$$\frac{\mathbf{v}_{i,k+1}+\mathbf{v}_{i,k}}{2}=\frac{\mathbf{r}_{i,k+1}-\mathbf{r}_{i,k}}{\Delta t} \tag{20}$$

$$\mathbf{a}_{i,k}=\frac{\mathbf{v}_{i,k+1}-\mathbf{v}_{i,k}}{\Delta t} \tag{21}$$

If $\mathbf{F}_{i,k}$ is the force acting on P_i at time t_k, then force and acceleration are assumed to be related by

$$\mathbf{F}_{i,k}=m_i\mathbf{a}_{i,k} \tag{22}$$

If $\mathbf{F}_{i,k}$ is a central, "$1/r^2$" force, like gravitation or Coulombic interaction, then the arithmetic, conservative force formula (Greenspan, 1980a, p. 11) is

$$\mathbf{F}_{i,k}=\frac{\alpha(\mathbf{r}_{i,k+1}+\mathbf{r}_{i,k})}{r_{i,k}r_{i,k+1}(r_{i,k}+r_{i,k+1})}$$

More generally (Greenspan, 1980a, p. 31), if P_i interacts with the other $n-1$ particles and the force is attractive like $1/r^p$ and repulsive like $1/r^q$, then the arithmetic, conservative force on P_i is given by

$$\mathbf{F}_{i,k}=m_i\sum_{\substack{j=1\\ j\neq i}}^{n}\left[m_j\left(-\frac{G\left[\sum_{\xi=0}^{p-2}\left(r_{ij,k}^{\xi}r_{ij,k+1}^{p-\xi-2}\right)\right]}{r_{ij,k}^{p-1}r_{ij,k+1}^{p-1}(r_{ij,k}+r_{ij,k+1})}+\frac{H\left[\sum_{\xi=0}^{q-2}\left(r_{ij,k}^{\xi}r_{ij,k+1}^{q-\xi-2}\right)\right]}{r_{ij,k}^{q-1}r_{ij,k+1}^{q-1}(r_{ij,k}+r_{ij,k+1})}\right)(\mathbf{r}_{i,k+1}+\mathbf{r}_{i,k}-\mathbf{r}_{j,k+1}-\mathbf{r}_{j,})\right] \tag{23}$$

where $G\geqslant 0$, $H\geqslant 0$, $q>p\geqslant 2$, and $r_{ij,k}$ is the distance between P_i and P_j at time t_k.

Note that, with regard to the motion of a single particle, arithmetic conservative formulas are special cases of the following general formula (Greenspan, 1980a). For any Newtonian potential $\phi(r)$, let

$$\mathbf{F}_k=-\frac{\phi(r_{k+1})-\phi(r_k)}{r_{k+1}-r_k}\cdot\frac{\mathbf{r}_{k+1}+\mathbf{r}_k}{r_{k+1}+r_k} \tag{24}$$

Arithmetic formula (24) conserves exactly the same energy, linear and angular momentum as does its continuous, limiting counterpart

$$\mathbf{F}=-\left(\frac{\partial\phi}{\partial r}\right)\frac{\mathbf{r}}{r} \tag{25}$$

Observe, also, that only for simple forces, like gravity, do the continuous and the discrete approaches yield *exactly* the same dynamical behavior. In general (LaBudde and Greenspan, 1976), the two approaches yield results which differ by terms of order $(\Delta t)^3$.

Finally, note that a discrete conservative Hamiltonian theory has also been developed recently (LaBudde, 1980).

5. DISCRETE MODELS

A variety of particle, or discrete, models have been developed in the spirit of Section 3. These have been devised for simulations of string vibrations; heat conduction and convection; free surface, laminar and turbulent fluid flows; jet stream evolution; shock wave generation; elastic vibration; porous flow; stress wave propagation; planetary evolution; and biological self-reorganization (Greenspan, 1980a; 1981; Reeves and Greenspan, 1980). For illustrative purposes, we will now summarize several typical computer simulations and indicate, wherever possible, the derived insights and advantages.

Because of the almost omnipresence of fluids, that is, of liquids and gases, let us begin by examining a completely conservative fluid model (Greenspan, 1974a) which utilizes the formulas of Section 4. Figure 1 shows how particles of a liquid emerge from a nozzle at relatively low speeds. This flow is what is usually called laminar. As the particle velocities are increased moderately, the rows of particles maintain their relative positions, as shown in Figure 2, but the flow is becoming chaotic. The disturbances arise because the increase in speed brings particles closer to each other and induces relatively large repulsive forces. Finally, in Figure 3, the speeds have been increased to the point where the rows no longer maintain their relative positions, and the motion is called turbulent. Indeed, if a vortex is defined as a set of particles rotating together in either a clockwise or a counterclockwise direction, then what we have called turbulent flow exhibits the rapid appearance and disappearance of vortices, which is, indeed, the engineering rule-of-thumb definition. It is interesting to note that our model allows for transition from laminar to turbulent flow merely by an increase in speed. In contemporary continuum mechanics, not only is this not possible, but there is, as yet, no realistic model of turbulence (Saffman, 1968).

As a second application of the conservative formulas of Section 4, Figure 4 shows the elastic vibration of a flexible bar from a position of tension (Greenspan, 1974b). What emerges clearly is that the bar does not swing "smoothly," but flutters up, due to waves which travel through the bar as part of its gross upward motion.

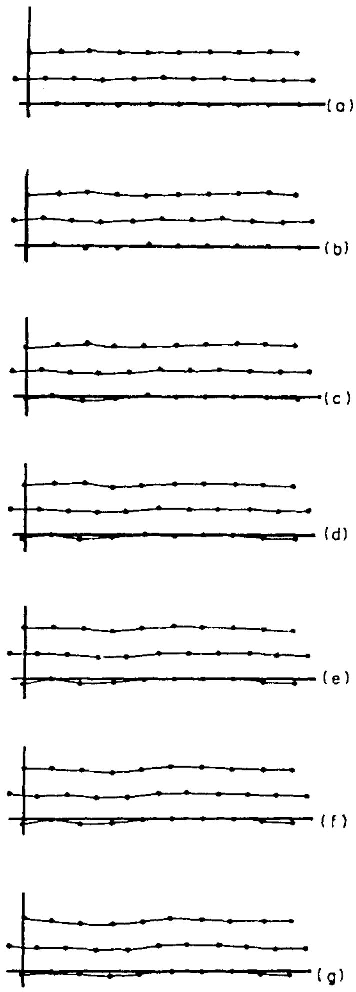

Fig. 1. Laminar flow.

With regard to the examples above, note that the computer implementation of conservative modeling is relatively expensive, in the sense that the dynamical difference equations are implicit and, therefore, require the solution of a system of nonlinear algebraic equations at each time step. However, without sufficient resources, one can still continue in the spirit of the modeling, as described in Section 3, by using explicit difference equations (Greenspan, 1980a). In this fashion, economy is gained at the expense of exact conservation. We will describe next several models developed using

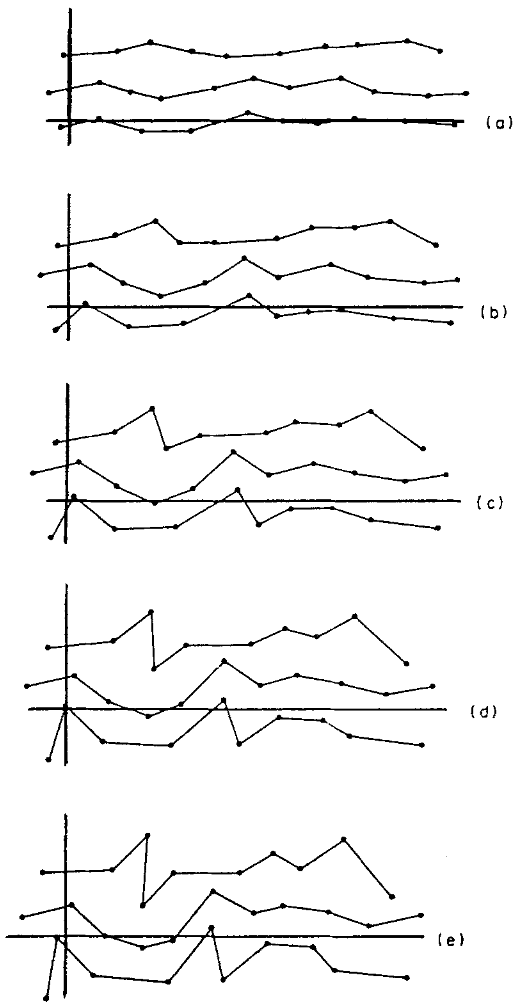

Fig. 2. Chaotic flow.

the explicit leap-frog formulas. It should be remembered, however, that this modeling could have been done also with the conservative formulas of Section 4.

Figure 5 shows the entry and dispersion of a liquid drop into a liquid well, the accompanying free surface wave generation, and the flow over the right wall (Greenspan, 1980b). The drop particles, which have been darkened in the figure, are shown initially at the time when the drop has flattened and is ready to enter into the well. The fluid motions are analyzed easily, as

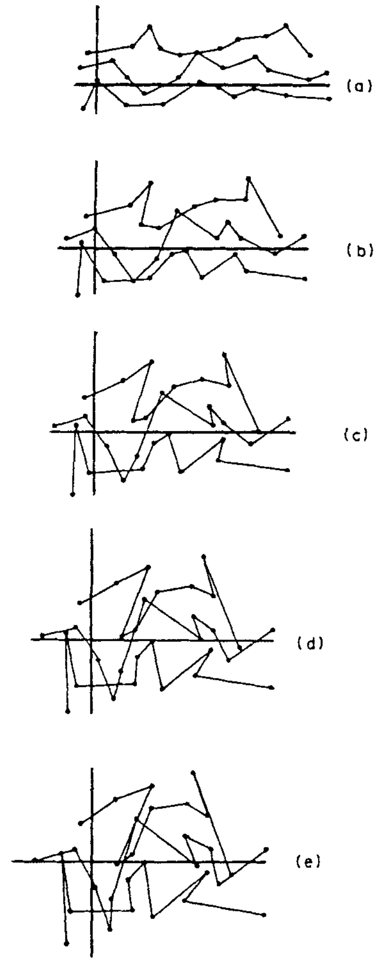

Fig. 3. Turbulent flow.

shown in Figure 6, by following the behavior of various liquid columns. In this fashion the gross behavior deduced is that particles which were originally on the fluid surface move to the right, while those below the surface have come up to the left to cover the sinking drop particles. Extension of this example to three dimensions is straightforward and yields completely analogous results.

Figures 7 and 8 show two sequential stages in the evolution of a lunar-type body from a hot, swirling gas (Greenspan and Collier, 1978). This model includes heating from a sun and heat radiation from the dark

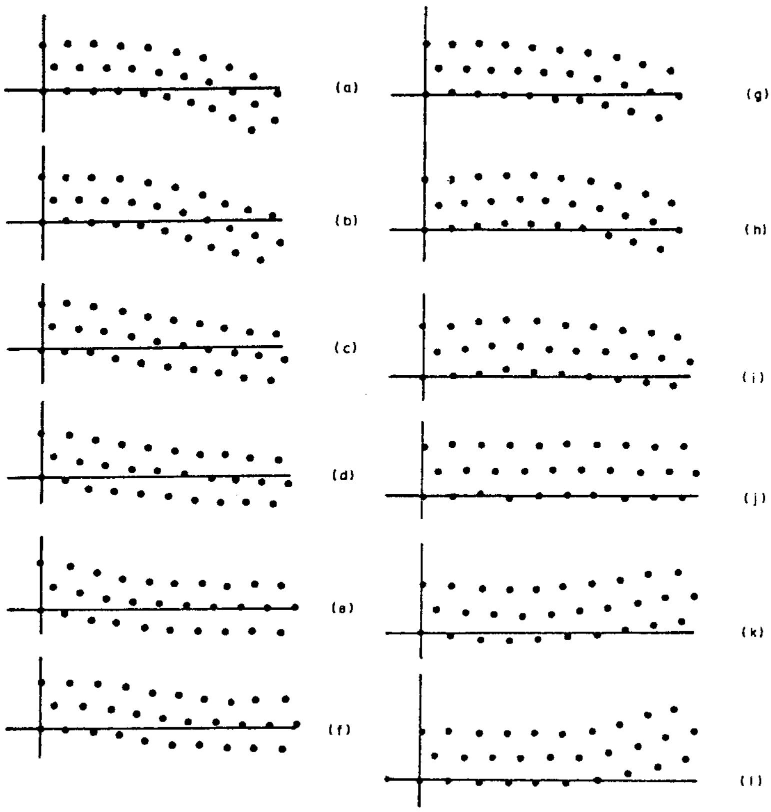

Fig. 4. Vibration of a bar.

side of the body. Figure 8 shows solidified (hexagonal) particle formations interior to the body and on the surface, in addition to liquid particles both on the surface and below the surface. Moreover, the figures show that particle modeling allows for self-reorganization. In Figure 7 one sees that the heavier particles have organized into two groups, and then, in Figure 8, that the two groups have moved towards each other.

The self-reorganization capability of particle modeling was also applied to an area of biology which is concerned with cell sorting. Recent experiments (Steinberg, 1963), for example, show that when tissue mesoderm, endoderm, and ectoderm cells are separated, the cells self-reorganize into the original mesoderm, endoderm, and ectoderm configuration. Figure 9 shows a particle model (Greenspan, 1981) of such a self-reorganization.

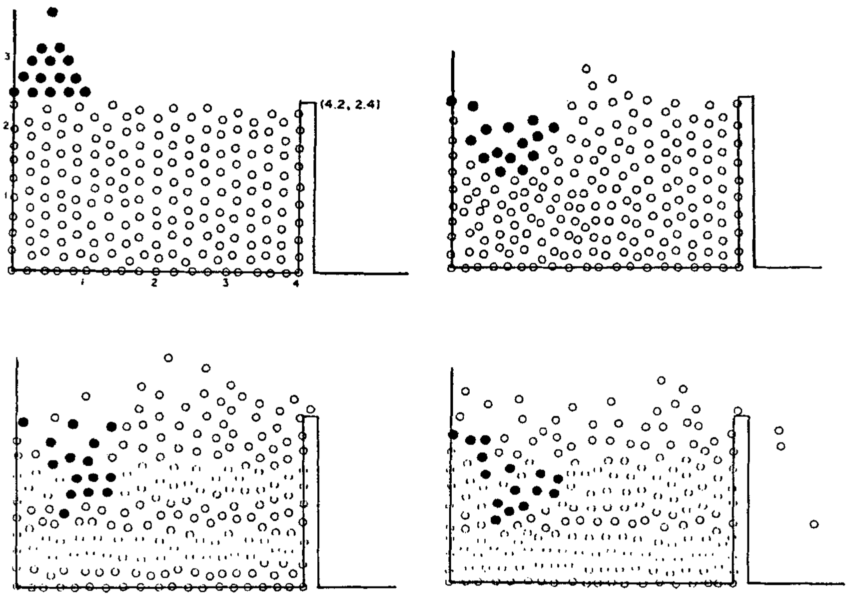

Fig. 5. Fluid drop dispersion and free surface wave generation.

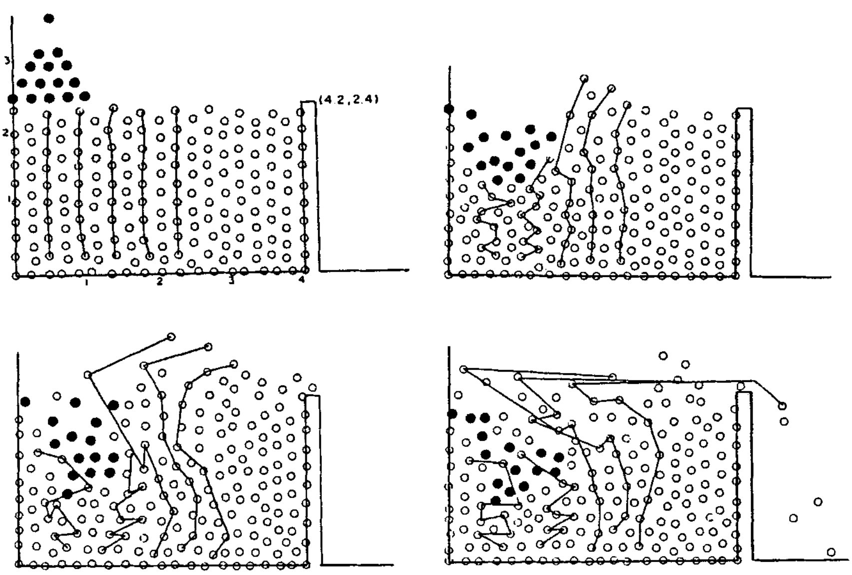

Fig. 6. Fluid drop analysis.

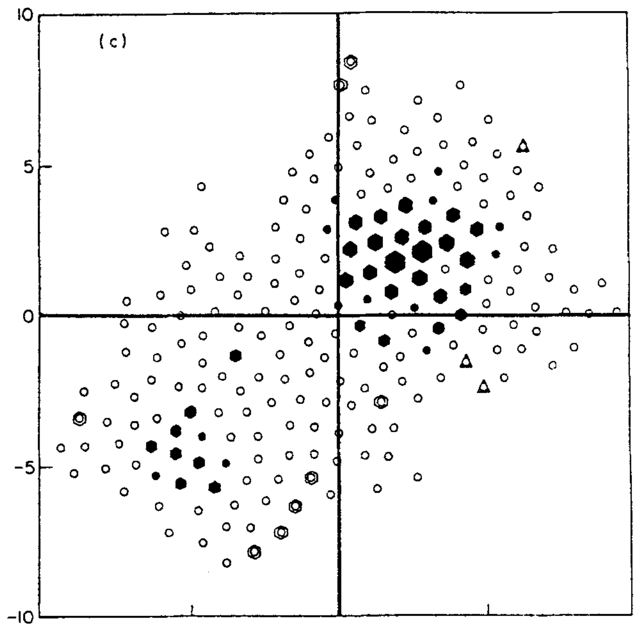

Fig. 7. Fluid phase of planetary evolution.

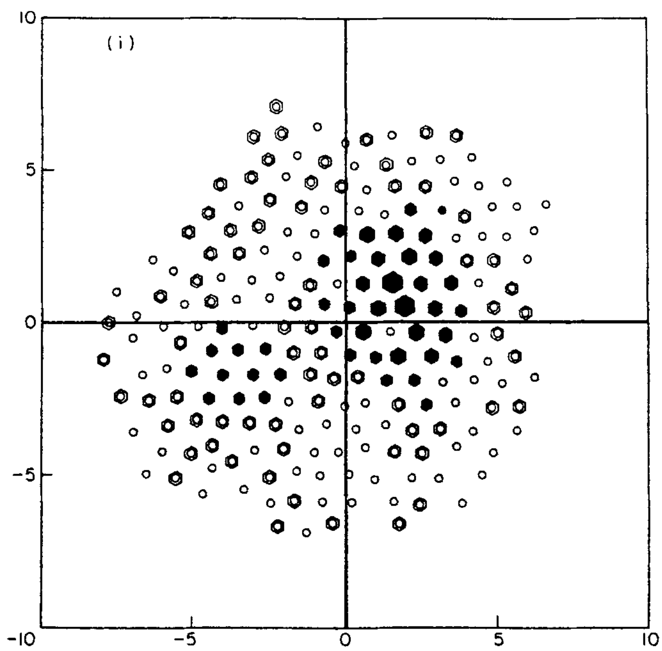

Fig. 8. Fluid-solid phase of planetary evolution.

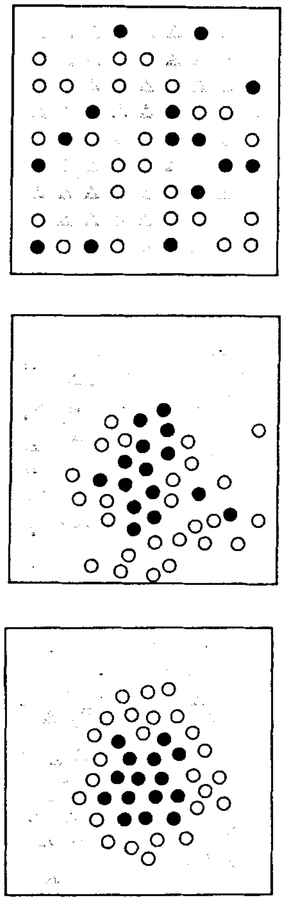

Fig. 9. Biological cell sorting.

We note finally that in each of the models developed thus far, the behavior has been described qualitatively. However, inclusion of experimental data into the force formulas has led recently to a quantitatively accurate model (Reeves and Greenspan, 1980) of stress wave propagation in both tapered and untapered aluminum bars.

6. SPECIAL RELATIVITY

Next, let us show that an arithmetic basis exists for the other broadly accepted, deterministic theory of mechanics, that is, special relativity (Greenspan, 1980a). For simplicity, we will do this in one space dimension only and for variety will emphasize the symmetry property rather than energy conservation, which was emphasized for Newtonian mechanics. The extension to more than one dimension, the establishment of the conservation laws, and the proof that $E=mc^2$ all follow from the basic numerical formulas to be given and from the arithmetic analogs of various relativistic concepts.

Consider, then, two Euclidean coordinate systems XYZ and $X'Y'Z'$ which at some initial time coincide. Let $X'Y'Z'$ be the rocket frame and let it be in constant uniform motion relative to lab frame XYZ. Assume that the constant relative speed is u and that the axes Y and Y' are always parallel, as are the axes Z and Z'. For $\Delta t>0$, an observer in the lab frame makes observations at the distinct times $t_k=k\Delta t$, $k=0,1,2,\ldots$. Using an identical, synchronized clock, an observer in the rocket frame makes observations at the times t'_k, where t'_k on the rocket clock corresponds to t_k on the lab clock.

Now, if particle P is at (x_k, y_k, z_k) in the lab frame at time t_k, while it is at (x'_k, y'_k, z'_k) in the rocket frame at time t'_k, then the variables are related by the Lorentz transformation

$$x'_k=c(x_k-ut_k)/(c^2-u^2)^{1/2} \tag{26}$$

$$y'_k=y_k \tag{27}$$

$$z'_k=z_k \tag{28}$$

$$t'_k=(c^2t_k-ux_k)\big/\left[c(c^2-u^2)^{1/2}\right] \tag{29}$$

where c is the speed of light.

In the lab frame, let particle P be in motion in the X direction. Then, at time t_k, P's velocity $v(t_k)=v_k$ and acceleration $a(t_k)=a_k$ are defined by

$$v_k=\Delta x_k/\Delta t_k \tag{30}$$

$$a_k=\Delta v_k/\Delta t_k \tag{31}$$

where the forward difference operator Δ is defined in the usual way by

$$\Delta f(t_k)=f(t_{k+1})-f(t_k)$$

In the rocket frame, at time t'_k, one defines v'_k and a'_k by

$$v'_k=\Delta x'_k/\Delta t'_k \tag{32}$$

$$a'_k=\Delta v'_k/\Delta t'_k \tag{33}$$

In order to find the relationship between v_k and v'_k, and between a_k and a'_k, note that (26)–(29) imply

$$\Delta x'_k = c(\Delta x_k - u\Delta t_k)/(c^2 - u^2)^{1/2} \tag{34}$$

$$\Delta y'_k = \Delta y_k \tag{35}$$

$$\Delta z'_k = \Delta z_k \tag{36}$$

$$\Delta t'_k = (c^2\Delta t_k - u\Delta x_k)/\left[c(c^2 - u^2)^{1/2}\right] \tag{37}$$

Thus, (32), (34), and (37) imply

$$v'_k = \left[c^2(v_k - u)\right]/(c^2 - uv_k) \tag{38}$$

while (33), (37), and (38) imply

$$a'_k = \frac{c^3(c^2 - u^2)^{3/2}}{(c^2 - uv_k)^2(c^2 - uv_{k+1})} a_k \tag{39}$$

Next, we assume that the mass m of particle P depends on its velocity in the following way. In the lab frame, the mass m_k of P at time t_k is assumed to satisfy the relationship

$$m_k = cm^*/(c^2 - v_k^2)^{1/2} \tag{40}$$

while its mass m'_k at the corresponding time t'_k in the rocket frame is assumed to satisfy

$$m'_k = cm^*/(c^2 - v_k'^2)^{1/2} \tag{41}$$

In (40) and (41), m^* is the rest mass of P and both $|v_k|$ and $|v'_k|$ are assumed to be smaller than c.

We now come to the problem of interest. From the dynamical point of view, the actual motion of a particle in, say, the lab frame can be determined from (30) and (31) once an equation which relates force and acceleration is given. We will take this equation to be

$$F_k = \frac{c^2 m_k}{\left[(c^2 - v_k^2)(c^2 - v_{k+1}^2)\right]^{1/2}} \cdot \frac{\Delta v_k}{\Delta t_k} \tag{42}$$

However, by the principle of relativity, in the rocket frame one must have symmetry, that is

$$F'_k = \frac{c^2 m'_k}{\left[\left(c^2 - v_k'^2\right)\left(c^2 - v_{k+1}'^2\right)\right]^{1/2}} \cdot \frac{\Delta v'_k}{\Delta t'_k} \tag{43}$$

But this is valid only if the right-hand side of (43) maps into the right-hand side of (42) under the Lorentz transformation. Fortunately, this is correct, since

$$\frac{c^2 m'_k}{\left[\left(c^2 - v_k'^2\right)\left(c^2 - v_{k+1}'^2\right)\right]^{1/2}} \cdot \frac{\Delta v'_k}{\Delta t'_k} = \frac{c^3 m^*}{\left(c^2 - v_k'^2\right)\left(c^2 - v_{k+1}'^2\right)^{1/2}} a'_k$$

$$\frac{c^3 m^*}{\left(c^2 - v_k^2\right)\left(c^2 - v_{k+1}^2\right)^{1/2}} a_k = \frac{c^2 m_k}{\left[\left(c^2 - v_k^2\right)\left(c^2 - v_{k+1}^2\right)\right]^{1/2}} \cdot \frac{\Delta v_k}{\Delta t_k}$$

Note that taking limits in (42) yields the particular form

$$F = \frac{c^2 m}{c^2 - v^2} \cdot \frac{dv}{dt}$$

of the classical Einstein equation

$$F = \frac{d}{dt}(mv), \qquad m = cm^*/\left(c^2 - v^2\right)^{1/2}$$

Note also that use of (42) in the lab frame and of (43) in the rocket frame implies that the numerical results are related by the Lorentz transformation. That is, if one were to install identical computers in the lab and rocket frames and use force laws (42) and (43), in the respective frames, then all resultant computations would be related by the Lorentz transformation. Such calculations have been carried out (Greenspan, 1980a) for a relativistic harmonic oscillator.

7. REMARKS

Let us note first that at a time when instruments for measurement, like electron microscopes, atomic clocks, and radio telescopes, are indicating fundamental nonlinear behavior in natural phenomena, it is of value to have particle modeling available, since it is fully nonlinear.

Next, note that with the development of parallel computation technology, we expect to be able to study discrete models which have, approximately, 100000 particles with relative ease and minimal expense.

Finally, let us indicate the fundamental difference between discrete and continuum modeling. Suppose one has a glass of water, in which there are, say, 10^{30} molecules. In continuum mechanics, we use the approximation $10^{30} \sim C$, where C is that infinite number which represents the cardinality of the continuum. In discrete modeling, we use the approximation, say, $10^{30} \sim 10^{3}$ and simultaneously compensate by adjusting molecular parameters. Since our conditioning usually leads us to feel more comfortable with the continuum approach, the following should be observed. In studying the motion of 10^{30} molecules, we need be concerned only with the motions of their centers of mass. These 10^{30} points, however, form a set of measure zero in C points, or, less mathematically, 10^{30} points are lost entirely in C points, so vast is the continuum. The lesson to be learned is that both discrete and continuum models are, indeed, only models, or approximations, of the real thing.

REFERENCES

Greenspan, D (1973). *Discrete Models*. Addison-Wesley, Reading, Massachusetts.

Greenspan, D. (1974a). "An Arithmetic, Particle Theory of Fluid Dynamics," *Computational Mathematics and Applied Mechanical Engineering*, **3**, 293–303.

Greenspan, D. (1974b). "Discrete Bars, Conductive Heat Transfer, and Elasticity," *Computers and Structures*, **4**, 243–251.

Greenspan, D. (1980a). *Arithmetic Applied Mathematics*. Pergamon, Oxford.

Greenspan, D. (1980b). "*N*-Body Modeling of Nonlinear, Free Surface Liquid Flow," *Mathematics and Computation in Simulation*, **XXII**, 200–207.

Greenspan, D. (1981). "A Classical Molecular Approach to Computer Simulation of Biological Sorting," *Journal of Mathematical Biology*, **12**, 227–235.

Greenspan, D., and Collier, J. (1978). "Computer Studies of Swirling Particle Fluids and the Evolution of Planetary-Type Bodies," *Journal of the Institute for Mathematics and its Applications*, **22**, 235–253.

LaBudde, R. A. (1980). "Discrete Hamiltonian Mechanics," *International Journal of General Systems*, **6**, 3–12.

LaBudde, R. A., and Greenspan, D. (1976). "Energy and Momentum Conserving Methods of Arbitrary Order for the Numerical Integration of Equations of Motion—II," *Numerische Mathematik*, **26**, 1–16.

Reeves, W. R., and Greenspan, D. (1980). "An Analysis of Stress Wave Propagation in Slender Bars Using a Discrete Particle Approach," TR 142, Department of Mathematics, University of Texas Arlington, 1980, to appear in *Applied Mathematical Modelling*.

Saffman, P. G. (1968). "Lectures on Homogeneous Turbulence," in *Topics in Nonlinear Physics*. Springer-Verlag, New York, pp. 485–614.

Steinberg, M. S. (1963). "Reconstructing of Tissues by Dissociated Cells," *Science*, **141**, 401–408.